Lessons from turbulence

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Introduction

It is trite to regard turbulence as the "last unsolved problem" in classical physics and to cite many books on the subject, as well as various authorities in Physics, to justify the opinion. It is likewise traditional to list a number of great physicists and mathematicians, such as Werner Heisenberg, Richard Feynman and Andrei Kolmogorov, who "failed" to solve the problem despite much effort, and thus caution youngsters that it is a difficult problem beyond reasonable grasp. Horace Lamb is credited with wishing to seek heavenly wisdom on the subject when he arrived there. (This story has been attributed variously to others such as Heisenberg and Sommerfeld.) Where do we stand after all these years? Do we adopt these same standard musings and apology for this article? Have we learnt something of value, both internal to the subject itself and in relation to neighboring problems in Physics?

Turbulence has indeed contributed several tools and ideas of lasting value to neighboring areas of Physics. An incomplete list includes negative temperature, anomalous diffusion and the very notion of power-law scaling in many-body problems. The powerful notions of scale-invariance and universality were first proposed in turbulence. In addition, turbulence is the playground of solutions that are non-unique or depend sensitively on initial conditions, and one for which the problem of predictability was posed for the first time in concrete terms. It is also the physical problem for which the coexistence of structure and randomness was recognized in most unambiguous terms, as was the role of correlated structures in transport of matter, heat and momentum. Internally, the subject has undergone tremendous changes. For instance, a graduate course on turbulence today would be quite different from that taught some thirty years ago.

Of course, turbulence is not a single problem but rather a huge field with different applications in engineering, geophysics, astrophysics and cosmology; it is also an excellent source of problems for pure mathematics. Much work has been done by applied mathematicians, meteorologists and engineers, in which the focus is often on particulars such as drag and pressure drop, mean velocity distributions, mixing efficiencies and dispersion rates. Indeed, by considering the totality of these problems as "turbulence", one can justify the statement that the problem remains unsolved, perhaps just as the structure of complex atoms and molecules remains unsolved from the standpoint of fundamental principles of quantum mechanics.

On the other hand, the diversity of problems in turbulence and their specific complexities should not obscure the fact that the heart of the subject belongs to Physics—the central problem being that of strong fluctuations and strong coupling, as in field theory and condensed matter physics. The recent progress made in this specific context has been substantial, and it is now possible to understand a few fundamental properties of turbulence. In particular, we shall present here the new concept of statistical conservation laws and describe their role in the breaking of symmetries of the turbulent state. This enables a powerful look at the concept of universality in turbulence which, perhaps ironically, highlights its limitations. We believe that the lessons learnt from this recent work are of general applicability in equilibrium and far-from-equilibrium systems.

The inertial range

For our purposes here, the paradigm of turbulence is the flow of fluids, though the concept can be generalized to solids and plasma. Turbulent motion is irregular in both time and space. It can be sustained against dissipation only by an external body force, or the addition of energy at the boundary, without which it decays eventually. A general aim is to understand the common features of systems that have many strongly interacting degrees of freedom and are far from equilibrium.

An examination of a fluid confined in a tank but set in motion by an external stirrer can usefully get across some basic ideas. Consider a tank of fluid in which a swirling blade of diameter L, which is comparable in size to the linear dimension of the tank, stirs the flow with a characteristic velocity $V \equiv \Omega L/2$. The resulting motion of the fluid can be expected to depend on the density and viscosity of the fluid, in addition to V and L. However, since the fluid density changes little within the flow, it can be regarded as a constant, and all properties, including the power supplied to the blade, can be normalized by that density—thus eliminating it from consideration. The nature of the boundaries of the tank and the blade could play some role, but we shall focus on the state of the flow far from boundaries and regard that these aspects enter only benignly, if at all. Then the only features that affect the flow are L and V and the kinematic viscosity ν of the fluid. From these physical parameters, one can construct the non-dimensional parameter $Re \equiv VL/\nu$, called the Reynolds number. Because viscosity is an internal friction, it serves to smooth out variations in the velocity so that the flow will be smooth when Re is small. On the other hand, when Re becomes large (how large is large depends on the details of the flow configuration), the motion set up by the blade, whose scale is of the order L, will be dominated by nonlinear effects whose nature is contained in the governing equations. The nonlinearity produces motions of smaller scales, which, in turn, produce even smaller scales. Eventually, a hierarchy of scales of motions appears in the flow; turbulence in the steady state is the collective motion of these many scales.

The generation of smaller scales occurs more and more rapidly as the scale size decreases, which is why it is figuratively described as a "cascade" process. The cascade is terminated by viscosity at a characteristic scale η that is much smaller than L. Increasing Re, one makes η smaller thus increasing the gap between L and η. By being distant from both L and η, the intermediate scale range may be expected to be indifferent to both how energy is "injected" at the scale L and how it is dissipated at η. This range of scales is called the inertial range. According to an idea of Richardson [1], the kinetic energy that is injected at scales of the order L is cascaded without loss through the inertial range, and is dissipated by viscous action at scales of the order η. Indeed, this nonlinear energy transfer process is a dominant feature in the inertial range.

Because of the conceptual simplicity of the inertial range, it is natural to ask if our expectation of universality—that is, freedom from the details of external forcing and internal friction—is true at the level of a physical law. Another facet of the universality problem concerns features that are common to different turbulent systems. This quest for universality is motivated by the hope of being able to distinguish general principles that govern far-from-equilibrium systems, similar in scope to the variational principles that govern thermal equilibrium. Since all dynamical features of turbulence are irregular, the questions can be posed and answered only in terms of statistical averages—computed for appropriate regions of space, intervals of time, or suitably defined ensembles.

Scale invariance and universality

It is believed that all important properties of turbulence are contained in the Navier-Stokes equations for the fluid motion which can be written for unit density as

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \mathbf{f} + \nu \Delta \mathbf{v}\,, \tag{1}$$

supplemented by the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. Here $\mathbf{v}$ is the velocity vector, p is the local pressure, f is a stochastic forcing and the suffix t stands for time-derivative.

The forcing term f abstracts the effect of boundary conditions on obstacles or other ways of generating turbulence.

To study the motion in the inertial range, it is convenient to define the so-called structure functions which are moments of velocity differences δv_r across an inertial-range scale of size r. Structure functions of order n are defined as $S_n \equiv \left\langle [(\mathbf{v}(\mathbf{r},t) - \mathbf{v}(\mathbf{0},t)) \cdot \mathbf{r}/r]^n \right\rangle \equiv \left\langle (\delta v_r)^n \right\rangle$, with $r \equiv |\mathbf{r}|$. For example, the second-order structure function is the energy contained in all Fourier modes with wavenumbers larger than $1/r$. The third-order structure function has a special place because Kolmogorov [2] was able to derive, exactly from (1), the flux-constancy relation in the inertial range

$$S_3/r = -(4/5)\langle\epsilon\rangle. \tag{2}$$

This is known as Kolmogorov's four-fifths law. Here $\langle\epsilon\rangle = \nu\langle|\nabla\mathbf{v}|^2\rangle$ is the average rate of energy dissipation per unit mass, which can be identified, through this relation, with the energy flux across scales. For later purposes we note that, since the velocity changes sign under a time-reversal operation, the nonzero value of S_3 implies the breakdown of time-reversibility in the inertial range.

No one has yet been able to deduce moments of other orders similarly explicitly [3], but Kolmogorov [2] attempted another giant step. He used the assumption of scale invariance—by which is meant that the symmetry broken by large-scale forcing is restored at much smaller scales $r << L$—and concluded that the structure functions are power laws with exponents ζ_n that depend linearly on the moment order; that is, $S_n \propto r^{\zeta_n}$ with $\zeta_n = n/3$, consistent with the exact result (2) for the specific case of $n = 3$.

If Kolmogorov's scale invariance were exact, its practical consequences would be immense. For instance, many flows of practical interest cannot be computed directly from (1)—the airflow over an aircraft fuselage or that within a cloud has about 10^{18} excited degrees of freedom—and so have to be modelled in some way to be computationally feasible. This task would be relatively simple if the unresolved small scales could simply be rescaled via scale invariance.

Anomalous scaling

Modern evidence indicates that the scaling exponents of structure functions depart from the aesthetically appealing result presented above. The relative difference between the measured exponents and Kolmogorov's prediction is shown in figure 1, whose main message is that ζ_n is a nonlinear function of n. This breakdown of scale invariance in the inertial range, now called anomalous or multifractal scaling, is arguably an important feature of turbulence, and sets it apart from the usual critical phenomena: one needs to work out the behavior of moments of each order independently without succumbing to dimensional analysis. Anomalous scaling in turbulence is such that $\zeta_{2n} < n\zeta_2$ so that S_{2n}/S_2^n for $n > 2$ increases as $r \to 0$. The relative growth of high moments means that strong fluctuations become more probable as the scales become smaller. Its practical importance is that it limits our ability to produce realistic models for small-scale turbulence.

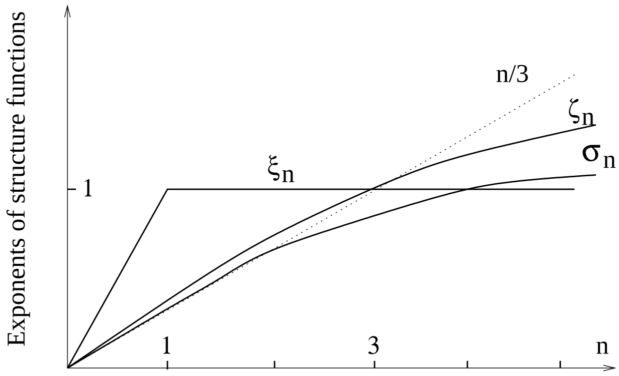

FIG. 1: The scaling exponents of the structure functions ξ_n for Burgers, ζ_n for 3d Navier-Stokes and σ_n for the passive scalar. The dotted straight line is $n/3$.

If scale invariance cannot be used to compute structure function exponents of all orders, what else is possible? Conservation laws impose constraints on the dynamics, and so conserved quantities (or their fluxes) play an essential role in answering this question. Energy conservation is broken in fluid turbulence at the large scale (through the addition of energy by stirring, as in figure 1) and at the scales of sharp gradients (through dissipation by fluid viscosity). Since neither feature dominates the inertial range, as seems plausible, energy conservation might be thought to apply there. However, as seen from (2), the basic dynamics of the inertial range is not the energy conservation (whose application gives the drastically unrealistic result of equipartition of energy over all scales), but the constancy of energy flux across scales from the large scales to small scales. In the steady state, the flux of energy equals the dissipation rate at small scales. For the first sight, it might appear from (1) that the dissipation rate of turbulent energy would vanish as $\nu \to 0$ (or as $Re \to \infty$), but an important feature of turbulence is that the average rate of energy dissipation per unit mass, $\langle\epsilon\rangle$, remains finite in this limit: no matter how small the viscosity, or how high the Reynolds number, or how extensive the scale-range participating in the energy cascade, the energy flux remains equal to that injected at the stirring scale. This is probably the first example of what is called "anomaly" in modern field-theoretical language: a symmetry of the inviscid equation (here, time-reversal invariance) remains broken even as the symmetry-breaking factor (viscosity) becomes vanishingly small [4]. If one screens a movie of steady turbulence backwards, we can tell that something is indeed wrong!

Recall from (2) that the third-order moment of turbulent velocity increments is determined completely by $\langle\epsilon\rangle$, whose finiteness is a consequence of the breakdown of time-reversal symmetry in the inertial range. Just as the breakdown of time-reversal symmetry, and the breakdown of energy conservation, govern the third-order structure function in the inertial range, are there other candidates for conserved quantities (or "integrals of motion") whose broken symmetries yield other structure functions? This is a fundamental question of turbulence—indeed of modern statistical physics—and is the basic thread that we shall pursue here. We shall first consider a relatively simple one-dimensional model.

Burgers turbulence

It was recognized already by von Neumann [5] that some important general features of turbulence can be understood by studying the equation suggested by Burgers [5]

$$u_t + uu_x = \nu u_{xx}. \tag{3}$$

The Burgers equation maintains some important properties of (1) and differs from it in others. In particular, it describes weakly compressible one-dimensional flows, as well as stochastic surface growth and other systems, and has been studied in much detail lately [5]. For zero viscosity, Burgers equation conserves an infinity of dynamical integrals of motion, $E_n = \frac{1}{2}\int u^{2n}\,dx$. For finite (and small) viscosity, however, this equation has as its solution a propagating shock wave. A random forcing that is correlated on large scales produces acoustic waves that evolve into shocks due to nonlinearity. This is the mechanism of energy cascade in Burgers turbulence. The shocks provide the mechanism for dissipation rates $\langle\epsilon_n\rangle$ appropriate to the integrals E_n. The shock dissipation has the property that the $\langle\epsilon_n\rangle$ of various orders tend to finite values as $\nu \to 0$, and it is easy to show that $S_{2n+1} = -4(2n+1)\langle\epsilon_n\rangle x/(2n-1)$ for all integer n. The simple scaling $S_{2n+1} \propto x$ can be readily appreciated since the probability of having a shock within the interval x is proportional to x, while the velocity difference across a shock is x-independent.

Note that the Burgers analog of (2) fixes $S_3 = -12\langle\epsilon\rangle x$ in a universal manner that is determined solely by $\langle\epsilon\rangle$, depending on neither the initial statistics (for the decaying case) nor the forcing (for the steady case). Scale invariance suggests that other structure functions $S_n(x)$ would be given by $(\langle\epsilon\rangle x)^{n/3}$ but this is not the case, as stated above. This fact means that the symmetry (scale invariance) broken by the forcing is not restored even when $x/L \to 0$, and that the small scales, however small, "remember" an infinity of input rates $\langle\epsilon_n\rangle$ determined by the forcing or initial conditions. We thus conclude that the breakdown of scale invariance in Burgers turbulence is related to an infinity of inviscid dynamical integrals of motion and to special structures (shocks) that are responsible for breaking the symmetry.

Statistical Lagrangian conservation laws (the "zero modes")

We now describe conservation laws that are qualitatively different. They are conserved only on the average, and determine the statistical properties of strongly fluctuating systems. While it is always possible to find in random systems fluctuating quantities with invariant averages, our question is more subtle: is it possible to find quantities that may be expected to change on dimensional grounds but stay constant nevertheless? Let us characterize n fluid particles in a random flow by inter-particle distances R_{ij} (between particles i and j) as in figure 2. Consider homogeneous functions $f(R_{ij})$ of the inter-particle distances with a nonzero degree ζ, i.e. $f(\lambda R_{ij}) = \lambda^\zeta f(R_{ij})$. When all the distances grow on the average according to $< R_{ij}^2 > \propto t^a$, say, one expects on dimensional grounds that a generic homogeneous function will grow as $f \propto t^{a\zeta/2}$. Are there particular values of ζ for which one can build specific functions that are conserved on the average? As particles move in a random flow, the n-particle cloud grows in size but the fluctuations in the shape of the cloud decrease in magnitude. Therefore, one may look for suitable functions of size *and* shape that have the property of being conserved because the growth in size is compensated by the decrease of shape fluctuations. For the simplest case of Brownian diffusion, the time derivative of the mean of any function of distances between particles is the Laplacian of that function. Harmonic polynomials turn the Laplacian into zero (and can thus be called zero modes of the Laplacian), and so are conserved statistically. For Laplacian diffusion, the zero modes are polynomials in R^2 or t so that the scaling dimension $\zeta_n = 2n$ of the n-particle mode depends linearly on n because particles move independently. The scaling exponents of the zero modes are thus determined by the laws governing the decrease of shape fluctuations.

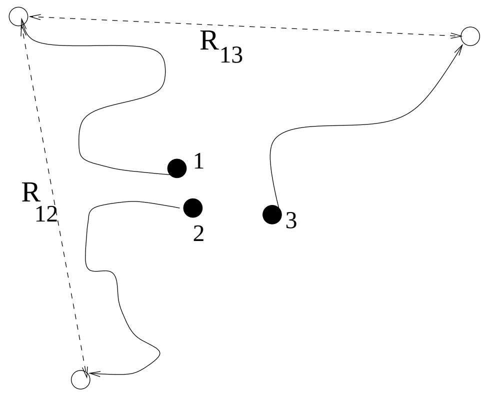

FIG. 2: Three fluid particles in a flow

The zero modes exist for turbulent diffusion as well but there is a major difference from the Brownian motion because the velocities of different particles are correlated in turbulence. Those mutual correlations make shape fluctuations decay more slowly than t^{-n} so that the exponents ζ_n of the zero modes grow with n more slowly than linearly. Indeed, power-law correlations of the velocity field lead to super-diffusive behavior of inter-particle separations: the farther the particles, the faster they tend to move away from each other, as in Richardson's law of diffusion. The system behaves as if there was an attraction between particles which weakens with the distance—though, of course, there is no physical interaction among particles but only mutual correlations (because they are contained within the correlation radius of the velocity field). We stress that while zero modes of multi-particle evolution exist for all velocity fields—from those that are smooth to those that are extremely rough as in Brownian motion—only those non-smooth velocity fields with power-law correlations in space generate anomalous scaling.

The importance of zero modes was discovered simultaneously by three groups [6] for the so-called Kraichnan model [7]. For this model, the statistically conserved quantities with an anomalous scaling were found as zero modes of the differential operator of the corresponding turbulent diffusion, and a comprehensive description of the theory can be found in [8]. The main ideas will be discussed below.

Anomalous scaling of scalar fields

We shall now ask: How does the existence of these statistical conservation laws (called martingales in the probability theory) lead to anomalous scaling of fields advected by turbulence? Consider first a passive scalar $\theta(\mathbf{r}, t)$, such as the temperature in a mildly heated flow, carried by the flow $\mathbf{v}(\mathbf{r}, t)$ in the presence of diffusive action due to κ, and sustained by an external forcing $\phi(\mathbf{r}, t)$, thus obeying

$$\theta_t + (\mathbf{v} \cdot \nabla)\theta = \phi + \kappa \Delta \theta. \tag{4}$$

If the correlation scale L_θ of the scalar forcing is much larger than the diffusion scale η_d determined by $\delta v_{\eta_d} \eta_d = \kappa$, one has a cascade of the scalar in the inertial range $\eta_d \ll r \ll L_\theta$, much like the energy cascade. The forcing produces large-scale fluctuations of the scalar field, which, through distortion by velocity gradients of increasingly smaller scales, develops increasingly smaller scales up to the point where diffusion smears them out at η_d. For scales larger than η_d, the correlation functions of θ are proportional to the times spent by the particles within L_θ. The structure functions of θ are differences of correlation functions with different initial particle configurations as, for instance, $S_3(r_{12}) \equiv \langle [\theta(\mathbf{r}_1) - \theta(\mathbf{r}_2)]^3 \rangle = 3\langle \theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2) - \theta(\mathbf{r}_1)\theta^2(\mathbf{r}_2) \rangle$. In calculating S_3, we are thus comparing two histories: the first one with two particles initially close to the position $\mathbf{r}_1$ and one particle at $\mathbf{r}_2$, and the second one with one particle at $\mathbf{r}_1$ and two particles at $\mathbf{r}_2$. That is, S_3 is proportional to the time during which one can distinguish one history from another, or to the time needed for an elongated triangle to relax to the equilateral shape. As can be seen from the right hand side of the equation for S_3, that time decreases as r_{12} grows; the further away the particles, the faster they loose correlations.

Quantitative details can be worked out when the velocity field v in (4), which is strictly speaking a solution of (1), is replaced by a stochastic field with an infinitesimally small correlation time but power-law correlations in space. This is the essential spirit of the Kraichnan model. Kraichnan's profound insight was that spatial rather than temporal correlations mattered for anomalous scaling. For this model, the scaling exponents of structure functions coincide with the scaling of the zero modes, and can be shown to follow $S_n(r) \propto r^{\zeta_n} L_\theta^{n(1-\alpha)-\zeta_n}$ [8]. Since ζ_n grows with n slower than linearly, we have $S_{2n}/S_2^n \propto (L_\theta/r)^{n\zeta_2 - \zeta_{2n}} \gg 1$, so that strong fluctuations of the scalar become more probable as $r/L_\theta \to 0$. Note that θ in the unforced and undamped case is transported unchanged and so is a Lagrangian invariant, as is the case of u in the inviscid and unforced Burgers equation. This means, in particular, that the integrals $\int \theta^n \, d\mathbf{r}$ are conserved (as for Burgers). However, the statistics of scalar field are determined by the statistical geometrical conservation laws rather than by the dynamical conservation laws (unlike in Burgers).

We thus see that the statistical conservation laws break the scale invariance of statistics in the inertial range, which explains why the scalar in the Kraichnan problem "knows" more about forcing than just the average flux of the variance. Further, if the forcing also breaks the isotropy of the scalar field, the anomalous scaling of the statistically conserved quantities will not restore isotropy in the limit $r/L_\theta \to 0$. This means that the probability density of scalar differences could get *more* anisotropic as $r/L \to 0$ [10].

It is worth emphasizing that scale invariance and isotropy (possibly broken by forcing) and time reversibility (broken by damping) are not restored even when $r/L \to 0$ and $\eta_d/r \to 0$, respectively. On the one hand, the anomalies produced by dynamically conserved quantities (like anomalous scaling in Burgers and time irreversibility in all cases of turbulence) are qualitatively different from the anomalies produced by statistically conserved quantities (like breakdowns of scale invariance and isotropy in passive scalar turbulence). Indeed, dissipation is a singular perturbation which breaks conservation of dynamical integrals of motion and imposes (one or many) flux-constancy conditions, very much similar to quantum anomalies [4]). On the contrary, there are no cascades of conserved quantity related to zero modes, nor their conservation is broken by dissipation. Anomalous scaling of zero modes is due to correlations between different fluid trajectories. On the other hand, the two types of anomalies are related intimately: the flux constancy imposes certain scaling properties of the velocity field, which generally leads to super-diffusion and to anomalous scaling of zero modes. Another important insight into the relation between anomalies and the non-smoothness of the velocity field goes back to Onsager's remark [11] that Lagrangian trajectories, which are the solutions of $d\mathbf{R}/dt = \mathbf{v}(\mathbf{R}, t)$, are not uniquely defined by initial conditions if the velocity field in the inviscid limit is non-differentiable. Thus, particles that start infinitesimally close together separate in a finite time. In particular, the structure functions involve initial configurations with just two groups of particles, as in $\langle \theta^2(\mathbf{r}_1)\theta(\mathbf{r}_2)\rangle$. For spatially smooth flows, the trajectories are unique so the particles which start together stay together. The only relevant degree of freedom then is the separation between the groups, which reduces the problem to two-particle dynamics. For non-smooth flows, particle separation means that we have distinct trajectories resulting from the evolution of n fluid particles that are not reducible to fewer particles.

One can also consider the case of $L \gg r \gg L_\theta$ which corresponds to thermal equilibrium with power-law correlations (a direct analog of critical phenomena). Here one can find an anomalously slow decay of correlations with the scale (as in quantum field theory and statistical physics), and identify the statistically conserved quantities responsible for anomalous scaling [12]. A consideration of passive scalars thus allows one to relate the breakdown of scale invariance to statistical integrals of motion both in turbulence and in equilibrium.

The results described have been derived for the Kraichnan model but there seems to be little doubt about their relevance for more general fields [8, 9]. The emphasis on particle trajectories has also brought a significant advance in numerical simulations of turbulence. By considering only a few trajectories (rather than the whole velocity field) one can model the phenomena that belong to the domain of high-Reynolds-number turbulence, such as anomalous scaling exponents; in particular, for the scalar field, the observed saturation of ζ_n at large n can be interpreted in terms of sharp fronts [13].

Two-dimensional turbulence

Large-scale motions in the atmosphere and shallow layers of fluid can be regarded as two-dimensional (2d) [14]. In 2d flows, the vorticity $\omega = \nabla \times \mathbf{v}$ is perpendicular to the velocity so that the stretching of vortex lines by the velocity field is absent. Indeed, (1) can then be written quite simply as

$$\omega_{\mathbf{t}} + (\mathbf{v} \cdot \nabla)\omega = f + \kappa \Delta \omega, \tag{5}$$

which implies that the vorticity is a Lagrangian invariant of the inviscid dynamics in the absence of forcing. As in the case of (3,4) with zero right hand sides, $\Omega_n \equiv \int \omega^n d\mathbf{r}$ are inviscid invariants. Among them, the squared vorticity, or enstrophy, $\Omega_2 = \int |\nabla \times \mathbf{v}|^2 d\mathbf{r}$, like the kinetic energy E, is arguably the most basic quantity. The existence of two quadratic and positive invariants, namely the kinetic energy and the enstrophy, means that there must be two cascades in the steady state of turbulence. Indeed, if we excite turbulence at some scale L injecting energy and enstrophy at finite rates ($\langle \epsilon \rangle$ and $\langle \varepsilon \rangle$, respectively), energy cannot cascade towards small scales because finite energy dissipation would mean infinite enstrophy dissipation in the limit of small viscosity. We thus come to the conclusion that energy must flow upscale while enstrophy should cascade downscale [15]. This particular consideration is a non-equilibrium development of Onsager's equilibrium treatment where joint conservation of energy and enstrophy lead to the notion of negative temperature [11]. Temperature is negative when the available phase volume *decreases* with energy, as happens at sufficiently high energy because finite enstrophy requires energy to be redistributed only among modes with lowest wave-numbers. The velocity spectrum in the inverse cascade (one of the most important results in the fifty years after Kolmogorov) was discovered by Kraichnan for two-dimensional incompressible turbulence and developed by Zakharov for wave turbulence [15].

What about other Ω_n? The intuition developed so far might suggest that the infinity of dynamical conservation laws must bring about anomalous scaling. Turbulence never fails to defy natural expectations. Consider first the direct cascade for which the constancy of the enstrophy flux takes the form

$$\langle(\mathbf{v}_1 \cdot \nabla_1 + \mathbf{v}_2 \cdot \nabla_2)\omega_1\omega_2\rangle = \langle\varepsilon\rangle \tag{6}$$

with subscripts referring to different locations. The relation (6) suggests the scaling of the velocity difference $\delta v_r \propto r$, or that the velocity field is smooth. Clearly, completely smooth velocity cannot provide for a finite vorticity dissipation in the inviscid limit, but (6) suggests that the logarithmic singularity will be enough. Indeed, particles separate exponentially in a smooth flow so the time of separation must behave as a logarithm of the distance. Therefore, were ω a passive scalar, it would have logarithmic correlation functions in a smooth velocity [16]. The fact that vorticity is not passive but related to velocity simply amounts to this: the field can be treated as a passive scalar, but the stretching rate acting on it must be renormalized with the scale [15, 17]. Experiments and numerics support the logarithmic behavior of correlation functions and the absence of anomalous scaling in the direct cascade [14].

For the inverse energy cascade where the analog of (2) is $S_3 = 3\langle\epsilon\rangle r/2$, the suggestion is that $\delta v_r \propto r^{1/3}$, as in three dimensional turbulence. An inter-particle distance stays within L during the time that is smaller, by a factor $(L/r)^2$, than that spent within r. If we now push (in a somewhat cavalier manner) the analogy between vorticity and passive scalar a bit further, we may assume that the vorticity correlation functions at $r > L$ will be proportional to these times and so must have a power-law form with exponents $-4n/3$, this being exactly the normal Kolmogorov scaling! Experiments indeed show normal scaling in the inverse cascade in two dimensions [14]. Moreover, a much more powerful conformal invariance has been discovered recently in the inverse cascade [18], which unites it with critical phenomena in two dimensions (in particular, iso-vorticity lines has been shown to have the same statistics as percolation clusters).

To conclude, the present knowledge is that 2d turbulence is scale-invariant at the scales both much larger and much smaller than the energy injection scale.

Vector fields

For the vector field (such as the velocity or magnetic field) the Lagrangian quantity that is statistically conserved may involve both the coordinate of the fluid particle and the vector that it carries. In general, the increase of the distance between particles can be offset by the decrease of correlations between the vectors they carry. Such conservation laws were built explicitly and related to the anomalous scaling of passively convected magnetic field for the Kraichnan model [8]. For vector fields, the two-point correlation function already involves the geometry and may be related to a nontrivial statistical integral of motion. As a result, the second-order velocity structure function is likely to have $\zeta_2 \neq 2/3$: the energy spectrum of the three-dimensional turbulence is thus not the celebrated $k^{-5/3}$ suggested by Kolmogorov, Obukhov, Onsager, Heisenberg and von Weizsäcker. It is somewhat ironic that probably the most widely known statement on turbulence, the so-called 5/3-rds spectrum, is exact not for three-dimensional turbulence for which it was initially proposed but for the inverse cascade in two dimensions.

Conclusions

Kolmogorov's scale-invariant theory, like Landau's theory of phase transition, had swayed the turbulence community for many years because of its beauty and simplicity. While the inadequacy of Kolmogorov's theory has been expected from a large body of recent experiments, the various caveats associated with the experiments had left room for questions. In critical phenomena, Onsager's solution of the Ising model played an important role—though it was not particularly highly regarded at the time—in convincing physicists that Landau's mean-field theory must be replaced. In this spirit, a search has been on for many years for the "Ising model" of turbulence, one in which the complexity of the problem is maintained in essential respects but is still solvable in a well-defined sense. Recent work on the Kraichnan model and Burgers turbulence have gone some way in this respect. One can now make a confident statement that stochastic differential equations of the turbulence-type indeed demonstrate the inadequacy of Kolmogorov's dimensional reasoning and simple closures. In particular, we have learnt that the new concept of statistical conservation laws plays a fundamental role in arriving at this conclusion. This is the key idea which we have attempted to explain in this article.

Briefly, the conserved quantities involve the geometry of multi-point configurations of fields advected by the flow. To provide for a cascade of the variance of these fields, the advecting velocity field must be non-smooth and generally possess power-law correlations in the inertial range. These properties produce correlations between fluid particles, which in turn make the scaling exponents of the statistical conservation laws nonlinear functions of the particle number. It turns out that the anomalous exponents of the statistically conserved quantities manifest themselves both in turbulent situations and in equilibrium.

It is our belief that the lessons just outlined are of universal validity, i.e. for other non-linear multidimensional systems which possess statistical conservation laws and anomalous exponents both in and out of equilibrium. In particular, it is our hope that future research will discover more fundamental links between turbulence, critical phenomena and other problems of condensed matter physics and field theory. In the end, "Physics [must be] well embedded in the seamless web of cross-relationships ..." [19].

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[2] A.N. Kolmogorov, *Dokl. Akad. Nauk. SSSR* **30**, 9-13 (1941); **32**, 16-18 (1941); reproduced in *Proc. Roy. Soc. A* **434**, 9-17 (1991).

[3] Equations for structure functions of all orders can be written down exactly from (1). Except for (2), they contain functions of higher orders and cannot be solved explicitly. This is the well-known "closure problem" in turbulence.

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